NIUS Lecture Notes

On

Waves & Solitons

Author:

Ajit Kumar

Indian Institute of Technology

Delhi

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PREFACE

Nonlinearity plays a very important role in science, be it physics, or chemistry, or biology, or economics, or any other descipline. Nonlinear revolution in physics took place primarily in the seventies and the eighties under the name of the discipline called "Chaos". By the chaotic behaviour of a dynamical system one means that the long-time behaviour of the system becomes unpredictable. Such a dynamics occurs in almost all physical systems that are under the action of an external force and are described by nonlinear and dissipative partial differential equations. For low intensity of the external force, the system behaves in the usual linear manner which we are familiar with in all branches of physics. However, as the intensity of the external force is increased, departure from the linear behaviour sets in and after the external force goes beyond a certain value, called the critical value, the dynamics of the system enters the so-called chaotic regime in which one can not predict the long-time behaviour of the system with any certainty whatsoever. The consequences are very interesting and have been studied in great details for a large variety of physical systems.

Another, may be more useful from the point of view of practical applications, manifestation of nonlinearity is the formation of the so-called solitons. A soliton is, in general, a localized travelling solitary wave solution of a nonlinear and dispersive (or diffractive) wave equation that interacts with another solitary wave elastically. Such excitations arise in almost all branches of physics where the physical system under investigation is weakly nonlinear and dispersive (or diffractive). Such objects have allowed physicists and engineers to explain many physical phenomena which could not be explained in the realm of linear physics. They have also been very useful for technological applications, especially, in nonlinear fiber optics where they are tipped to be the information carrying bits in an all-optical long-haul communication systems (see, for instance, References 3 and 6).

The present lectures are an endeavour to introduce undergraduate students to the concept called "Soliton". We start with the basics of travelling waves and their characteristics in a linear, non-dissipative and dispersionless system. Then we go on adding various perturbative terms, like dispersion, dissipation and nonlinearity, one by one to examine the changes introduced by them in the behaviour of the travelling wave. Solitary wave is introduced next and, finally, the concept of a soliton wave is presented. The lectures end with the discussion of the soliton solutions of some well known equations of physics. A small list of references is also given for any clarification which might be required.

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It's a great pleasure for me to extend my heartiest thanks to Professor Vijay Singh for inviting me to deliver these lectures at the NIUS Camp and the excellent hospitality at the HBCSE, Mumbai. I also thank Mr Praveen Pathak for technical assistance during the preparation of these lectures.

Ajit Kumar,

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Lecture 1 : Waves

Definition: A wave is a disturbance or variation that transports energy progressively from one point to the other in a medium.

Mathematically it is described by a second order linear partial differential equation, called the wave equation, which in one spatial dimension has the form

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \phi(x,t)}{\partial t^2} = 0, \tag{1}$$

where $\phi(x,t)$ stands for the deviation of a physical quantity (like, position, density, temperature, pressure, and electric field, depending on the problem at hand) from its equilibrium value. The quantity v is a constant and is called the wave velocity. This is the simplest form of the wave equation and does not take into account several important aspects, for instance, dispersion, dissipation, and nonlinearity. Dispersion causes waves of different frequencies to travel at different phase velocities, whereas, due to dissipation the amplitude of the wave goes on decreasing as it travels through the medium. Nonlinearity, on the other hand, leads to the steepening of the wave-front during propagation and, ultimately, leads to shock wave formation. As we shall see later the above wave equation can be modified appropriately to account for these aspects of waves.

Before moving forward, let us find the general solution of the above equation. Let us introduce new variables: $\xi = x - vt$ and $\eta = x + vt$. Then $\phi(x,t) = \phi(\xi,\eta)$ and

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi} + \frac{\partial \phi}{\partial \eta}, \qquad \frac{\partial \phi}{\partial t} = v\left(-\frac{\partial \phi}{\partial \xi} + \frac{\partial \phi}{\partial \eta}\right) \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial \xi^2} + 2\frac{\partial^2 \phi}{\partial \xi \partial \eta} + \frac{\partial^2 \phi}{\partial \eta^2}, \quad \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{v^2}\left(\frac{\partial^2 \phi}{\partial \xi^2} - 2\frac{\partial^2 \phi}{\partial \xi \partial \eta} + \frac{\partial^2 \phi}{\partial \eta^2}\right). \tag{3}$$

Therefore, the wave equation (1) reduces to

$$\frac{\partial^2 \phi(\xi,\eta)}{\partial \xi \partial \eta} = 0. \tag{4}$$

If we integrate equation (4) over ξ, we get

$$\frac{\partial \phi(\xi, \eta)}{\partial \eta} = C(\eta), \tag{5}$$

where $C(\eta)$ is an arbitrary function of its argument. Integrating now over η, we arrive at

$$\phi(\xi, \eta) = \int C(\eta) d\eta + f(\xi) \equiv g(\eta) + f(\xi). \tag{6}$$

Therefore, the general solution of the wave equation (1) is gioven by

$$\phi(x, t) = f(x - vt) + g(x + vt). \tag{7}$$

Here, $f(x - vt)$ and $g(x + vt)$ are arbitrary functions of their arguments. They represnt the so called d'Alembert solutions of the one dimensional wave equation and consist of a wave of constant shape (given by $f|_{t=0} = f_0(x)$) propagating along the positive x direction at a constant speed v and a wave of constant shape (given by $g|_{t=0} = g(x)_0$) propagating along the negative x direction at the same speed v.

To get convinced that it is really so, let us consider the wave $f(x-vt)$ propagating along the positive x direction. If we fix a point on this wave, corresponding to a fixed value ξ_0 of the argument $\xi = x - vt$, then it follows that such a point will move with a constant velocity $dx/dt = v$. If an observer runs along the positive x axis with a constant speed v, then in her frame

$$x' = x - vt, \quad t' = t, \tag{8}$$

and the wave will be given by $f(x')$. So, the observer will always see the same shape of the input wave form which in the stationary frame gets displaced along the positive x axis at a constant speed v.

Sinusoidal waves: One of the simplest wave forms is the sinusoidal one, given by

$$f(x, t) = A\, cos[k(x - vt) + \delta], \tag{9}$$

where A is a positive number and is called the wave amplitude. In the context of the wave on a stretched string, it represents the maximum displacement from the equilibrium position. The quantity δ is called the phase constant. The instantaneous snapshot of the wave is shown in Fig.1 and consists of an infinite series of indistinguishable troughs and crests.

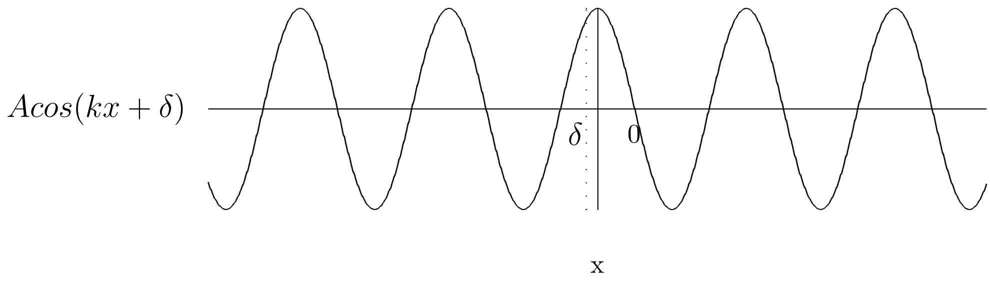

Fig.1: Instantaneous snapshot of the sinusoidal wave $f_{t=0} = Acos(kx + \delta)$.

Note that at $x = vt - (\delta/k)$, the argument of cosine becomes zero. Usually one calls it the central maximum. If $\delta = 0$, the central maximum passes the origin at $t = 0$. Therefore, for nonzero δ the quantity $-\delta/k$ gives the distance by which the central maximum, and hence the entire wave, is delayed. k is called the wave number and it is related to wavelength by the equation

$$k = (2\pi/\lambda), \tag{10}$$

because when x advances by $2\pi/k$ the cosine goes through one complete cycle.

As time passes, the entire wave train travels along the positive x direction at speed v. Once again if we refer to the wave travelling on an infinite stretched string, at a given x the string vibrates up and down and completes one full cycle in time

$$T = \frac{2\pi}{kv}. \tag{11}$$

This characteristic time is called the period of oscillation, since as time advances by this amount the cosine completes a full cycle. The number of oscillations per unit time is called the frequency and is given by

$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}. \tag{12}$$

The quantity $\omega = 2\pi\nu$ is called the angular frequency.

Complex notation: Using Euler's formula

$$e^{i\alpha} = cos\alpha + isin\alpha \tag{13}$$

the sinusoidal wave (7) can be written as

$$f(x,t) = Re\left(Ae^{i(kx-\omega t)}\right) \tag{14}$$

where $Re(\zeta)$ stands for the real part of ζ. Usually one introduces a complex notation for the travelling plane wave solution (see, for example, reference 1):

$$\tilde{f}(x,t) = \tilde{A}e^{i(kx-\omega t)}, \tag{15}$$

where $\tilde{A} = Ae^{i\delta}$ is the complex amplitude. The actual wave is given by $Re(\tilde{f}(x,t))$.

This wave given by (9) (or, equivalently by (15)) is called monochromatic because it involves a single frequency ν. This is also called plane wave because the displacement $\phi(x,t)$ from the equilibrium position depends only on one spatial coordinates and hence the wavefront, which is the locus of points all of which have the same phase at a given instant of time, is the plane perpendicular to the direction of propagation x. The wave is also undamped because the amplitude A is constant at all points along the direction of propagation. The energy of oscillation is proportional to the square of the amplitude A. Cosequently, the constancy of the amplitude implies that the energy is transferred from a point to another point without any losses. The intensity I of the wave, defined as the amount of energy transported by the wave per unit time across a unit area oriented normally to the direction of propagation at the point of observation, is also proportional to A^2.

One distinguishes between two kinds of waves: (i) Transverse wave and (ii) Longitudinal wave. When the displacement from the equilibrium is perpendicular to the direction of propagation the wave is called a transverse wave. On the other hand, if it is in the direction of propagation, the wave is called a longitudinal wave. In order to specify this property one assigns a new characteristic, called polarization, to the wave. It is denoted by a unit vector $\hat{n}$. Since there are two directions perpendicular to a given direction, transverse waves occur in two independent states of polarization. For instance, if we take the wave on a string to be propagating along the z direction, the vibrations of the string can occur either along the x direction or along the y direction. In the former case the wave is called to be x polarized and written as

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{x} \tag{16}$$

whereas in the latter it is called y polarized and written as

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{y}. \tag{17}$$

Clearly, we can have a more general case when the vibrations of the string are in a direction that makes an angle θ with the x axis. Then

$$\hat{n} = cos\theta\ \hat{x} + sin\theta\ \hat{y}. \tag{18}$$

The angle θ is called the angle of polarization and the wave is called a linearly polarized wave. It can be considered to be a linear superposition of two waves: one $\hat{x}$ polarized and the other $\hat{y}$ polarized. The complete wave function can be written as

$$\tilde{f}(z,t) = (\tilde{A}cos\theta)\ e^{i(kz-\omega t)}\ \hat{x} + (\tilde{A}sin\theta)\ e^{i(kz-\omega t)}\ \hat{y}. \tag{19}$$

Further, since the wave is a transverse one,

$$\hat{n}\cdot\hat{z} = 0. \tag{20}$$

Solution satisfying the initial conditions: Assume that the solution of the wave equation (1) exists and is given by Eq.(5). We want to determine the functions f and g such that the initial conditions

$$\phi(x,0) = \psi(x), \quad \dot{\phi}(x,0) = \theta(x) \tag{21}$$

are satisfied. The second condition, which physically represents the initial velocity, can be written as

$$v(g'(x) - f'(x)) = \theta(x) \tag{22}$$

Applying the above conditions, we obtain

$$f(x) + g(x) = \psi(x), \quad -f(x) + g(x) = \frac{1}{v}\int_{x_0}^{x} \theta(y)dy + C, \tag{23}$$

where x_0 and C are constants. Adding and subtracting the two equations (21), we have

$$f(x) = \frac{1}{2}\psi(x) - \frac{1}{2v}\int_{x_0}^{x} \theta(y)dy - \frac{C}{2}, \tag{24}$$

$$g(x) = \frac{1}{2}\psi(x) + \frac{1}{2v}\int_{x_0}^{x} \theta(y)dy + \frac{C}{2} \tag{25}$$

The required solution is, therefore, given by

$$\phi(x,t) = \frac{1}{2}[\psi(x - vt) + \psi(x + vt)] + \frac{1}{2v}\int_{x-vt}^{x+vt} \theta(y)dy. \tag{26}$$

Example: Consider a wave propagating along a stretched string which is initially at rest ($\phi(x,t)|_{t=0} = 0$). A velocity

$$\dot{\phi}(x) = Axe^{-x^2} \tag{27}$$

is then given to the string at $t = 0$. Determine the form of the wave at any $t > 0$. We have

$$f(x) + g(x) = 0, \quad -f(x) + g(x) = \frac{A}{v}\int_{0}^{x} ye^{-y^2}dy \tag{28}$$

Adding and subtracting the two equations, we obtain

$$f(x) = -\frac{A}{2v}\int_{0}^{x} ye^{-y^2}dy, \quad g(x) = \frac{A}{2v}\int_{0}^{x} ye^{-y^2}dy. \tag{29}$$

Therefore, the solution is given by

$$\phi(x,t) = \frac{A}{4v}\, e^{-(x-vt)^2} - \frac{A}{4v}\, e^{-(x+vt)^2}. \tag{30}$$

It consists of a gaussian travelling along the positive x direction and the same but inverted gaussian travelling along the negative x direction.

Lecture 2 : Electromagnetic waves. The wave packet.

One of the most important discoveries in physics was the theoretical prediction, by James Clerk maxwell, of the existence of displacement current in electrodynamics. It, in fact, completed the unification of electricity and magnetism, started by Faraday, who, on the basis of experimental observations, formulated the laws of electromagnetic induction. The set of basic differential equations, called Maxwell's equations, predicts the existence of electromagnetic waves even in free space. The present lecture is devoted to these waves.

The basic set of partial differential equations, that describes the spatio-temporal evolution of the electromagnetic fields in free space without sources, is given by

$$\vec{\nabla} \times \vec{B} - \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = 0, \quad \vec{\nabla} \cdot \vec{E} = 0 \tag{31}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \tag{32}$$

where $\vec{E}$ is the electric field intensity, $\vec{B}$ is the magnetic induction, and ε_0 and μ_0 are the permittivity and the permeability of the free space, respectively. They predict the existence of electromagnetic waves in free space. This is evident from the fact that Maxwell's equations admit three dimensional wave equation for the electric and the magnetic field vectors. If we take the *curl* of the first equation in (32), use the vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}^2 \vec{V}, \tag{33}$$

for an arbitrary vector field $V(\vec{r}, t)$ along with the equation (31), we arrive at the following partial differential equation satisfied by the electric field intensity $\vec{E}$

$$\nabla^2 \vec{E} - \varepsilon_0 \mu_0 \; \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{34}$$

Clearly, each of the electric field components E_i, $i = x, y, z$, satisfies the homogeneous wave equation which in the case of one spatial dimension reduces to the wave equation (1) of the previous lecture. The velocity of the wave is given by $1/\sqrt{\varepsilon_0\mu_0}$. The numerical value of the velocity turns out to be equal to the speed of light in vacuum. On the basis of this remarkable result Maxwell conjectured that light was an electromagnetic wave. This fact was then experimentally established by Hertz.

Note that the magnetic induction vector $\vec{B}$ also satisfies the same wave equation (34), however, it is customary to study electromagnetic phenomena in terms of the electric field intensity $\vec{E}$.

In a medium without sources the set of Maxwell's equations reads

$$\vec{\nabla} \times \vec{H} - \varepsilon_0 \frac{\partial \vec{D}}{\partial t} = 0, \quad \vec{\nabla} \cdot \vec{D} = 0 \tag{35}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \tag{36}$$

where $\vec{D} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon\vec{E}$ is the electric induction vector, $\vec{H} = \vec{B}/(\mu_0(1+\chi_m)) = \vec{B}/\mu$ is the magnetic field intensity, and ε and μ, respectively, are the permittivity and the permiability of the medium. The quantities χ_e and χ_m are respectively called the electric and the magnetic susceptibilities of the medium. Using the same procedure as earlier, we arrive at the following wave equation

$$\nabla^2 \vec{E} - \varepsilon\mu \ \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{37}$$

where

$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{n}, \tag{38}$$

$n = \sqrt{\varepsilon\mu/\varepsilon_0\mu_0}$ being the refractive index of the medium (see reference 1).

Plane wave solutions in free space: The plane wave solutions of the wave equation (37), are given by

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 \, e^{-i(\omega t - \vec{k}\cdot\vec{r})}, \quad \tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 \, e^{-i(\omega t - \vec{k}\cdot\vec{r})}, \tag{39}$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the complex amplitudes of the electric and the magnetic fields, respectively, $\vec{k}$ is the wave vector and gives the direction of propagation of the wave and $\omega = 2\pi\nu$ is the angular frequency. The physical fields are the real parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$.

Further, although the wave equation (37) was derived from the Maxwell's equations, every solution of the wave equation may not be the solution of the Maxwell's equations. The latter impose specific conditions on the solutions. Since $\vec{\nabla} \cdot \tilde{\mathbf{E}} = 0$ and $\vec{\nabla} \cdot \tilde{\mathbf{B}} = 0$, we have

$$\tilde{\mathbf{E}}_0 \cdot \vec{k} = 0, \quad \tilde{\mathbf{B}}_0 \cdot \vec{k} = 0. \tag{40}$$

The above result shows that the electric as well as the magnetic field does not have any component along the direction of propagation. That is, the electromagnetic wave is a transverse wave. Also, since

$$\vec{\nabla} \times \tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{B}}}{\partial t} = 0, \tag{41}$$

we get

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{k} \times \tilde{\mathbf{E}}_0) \tag{42}$$

It shows that the electric field and the magnetic field are in phase and mutually perpendicular. Their real amplitudes are related by

$$B_0 = \frac{E_0}{c}. \tag{43}$$

Energy and momentum of electromagnetic waves: Like any other mechanical system, electromagnetic waves carry energy, momentum and angular momentum. The energy per unit volume stored in the field is given by the electromagnetic energy density u_{em}

$$u_{em} = \frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right). \tag{44}$$

For a monochromatic wave propagating along the positive x direction (taking into account the real parts of the fields given by the equation (39)), we get

$$u_{em} = \varepsilon_0 \vec{E}^2 = \varepsilon_0 \vec{E}_0^2 cos^2(kx - \omega t + \delta), \tag{45}$$

and hence the total energy contained in a given volume V is

$$W_{em} = \int_V u_{em} d^3x \tag{46}$$

The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}). \tag{47}$$

Once again, for a monochromatic wave propagating along the positive x direction, we obtain

$$\vec{S} = c\varepsilon_0 \vec{E}_0^2 cos^2(kx - \omega t + \delta)\hat{x} = cu_{em}\hat{x}. \tag{48}$$

Since in time Δt a length $c\Delta t$ of wave passes through area A, carrying with it energy $cu_{em}A\Delta t$, the energy per unit area per unit time transported by the wave is, therefore, cu_{em}, which is what the previous formula represents. The wave also carries a momentum. The momentum density is defined as

$$\vec{p} = \frac{1}{c^2} \vec{S} = \frac{1}{c} u_{em}\hat{x}. \tag{49}$$

In the case of light, the wavelength is so short ($5 \cdot 10^{-7}$ m)and the period is so brief (10^{-15} s) that any macroscopic measurement will involve many cycles. Therefore, we talk about the average values of the above mentioned physical characteristics of light. Since the average value of $cos^2(kx - \omega t + \delta)$ over a cycle is half, we obtain:

$$\langle u_{em} \rangle = \frac{1}{2}\varepsilon_0 \vec{E}_0^2, \ \ \left\langle \vec{S} \right\rangle = \frac{1}{2} c\varepsilon_0 \vec{E}_0^2 \hat{x} = c \langle u_{em} \rangle \hat{x}, \ \ \langle \vec{p} \rangle = \frac{1}{2c}\varepsilon_0 \vec{E}_0^2 \hat{x} = \frac{1}{c} \langle u_{em} \rangle \hat{x}. \tag{50}$$

The average power per unit area transported by an electromagnetic wave is called its intensity I: $I = \langle S \rangle$.

Before we end this lecture, let us introduce the concept of a wave packet which represents a localized travelling wave solution of the wave equation. This will be useful when we introduce the concept of a solitary wave in the next lecture.

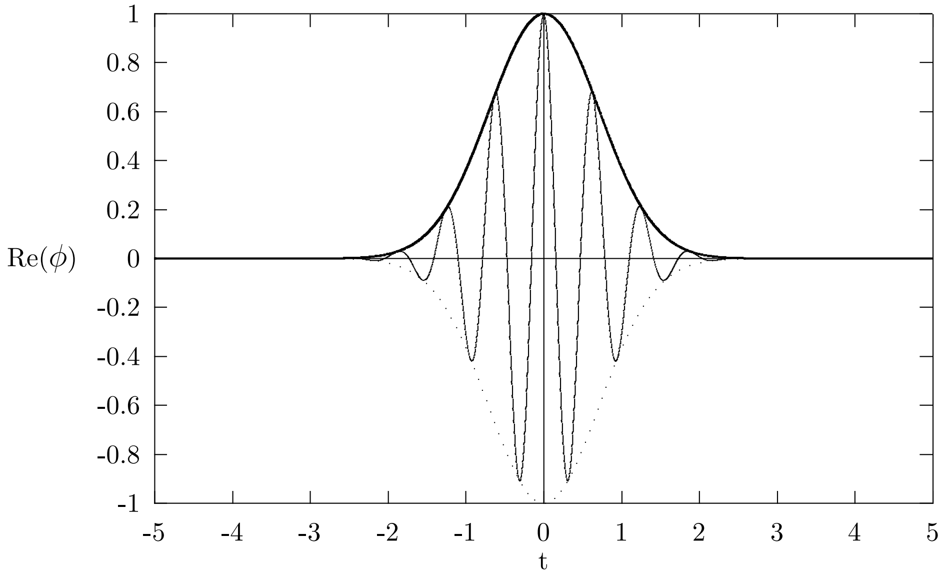

Fig.2 : A typical wave packet

The Wave packet: No source of electromagnetic radiation emits a single monochromatic wave. Even a laser light has a finite line width. What this means is that there exists a central (average) frequency of emission ν_0 (corresponding wave number is k_0) around which there are other waves with frequencies in the interval $(\nu_0 \pm \Delta\nu)$. Since the wave equation, we are dealing with, is a linear differential equation, any such solution, with the above mentioned frequency spread, can be represented as

$$\phi(x,t) = \int_{-\infty}^{+\infty} F(k)\, e^{i(kx-\omega(k)t)} dk, \tag{51}$$

where $F(k)$ is the amplitude of a Fourier component with wave number k. Negative k's have also been included to account for the waves propagating along the negative x direction. To understand the cosequences of such a superposition of monochromatic waves, let us consider the superposition of two harmonic waves, of the same amplitude A, propagating in the positive x direction

$$\phi_1(x,t) = Acos\left[(\omega_0 - \Delta\omega)t - (k_0 - \Delta k)x\right], \tag{52}$$

$$\phi_2(x,t) = Acos\left[(\omega_0 + \Delta\omega)t - (k_0 + \Delta k)x\right], \tag{53}$$

where $|\Delta k/k_0|$
$1, \Delta\omega/\omega_0 \ll 1$. The resultant wave is

$$\phi(x,t) = \phi_1(x,t) + \phi_2(x,t) = [2Acos(\Delta\omega\, t - \Delta k\, x)]\ cos(\omega_0 t - k_0 x). \tag{54}$$

The first factor inside the square brackets is a slowly varying function of x and t. Hence, the above equation can be regarded as the equation of a plane wave with slowly varying envelope amplitude $A_1 = 2Acos(\Delta\omega\, t - \Delta k\, x)$. The localized form of a typical wave packet is shown in Fig.2 at $x = 0$. As time passes the wave packet gets displaced along the x direction.

Within the limits of the packet the plane waves amplify one another to a greater or smaller extent. Outside these limits they virtually annihilate each other. Also, smaller the width of a packet the greater is the requency interval $\Delta\omega$ (or, equivalently, the interval Δk) needed to describe the wave packet because

$$\Delta k \Delta x \approx 2\pi \tag{55}$$

holds good for the wave packet.

The velocity of the center of the wave packet (it is the velocity at which the packet propagates along the x axis) is given by

$$v = \frac{\Delta\omega}{\Delta k} \tag{56}$$

which in the limit when $\Delta k \to 0$ becomes

$$v_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k} \tag{57}$$

and is called the group velocity of the wave packet. Note that v_g is always less than or equal to the speed of light: $v_g \leq c$ and gives the velocity at which the information, modulated on the wave, will propagate.

Lecture-3: Dispersive, dissipative and nonlinear wave equations and solitary waves

Let us see how we can modify the wave equation to account for effects like dispersion, dissipation and nonlinearity. It is quite legitimate to start with a wave $\phi(x-t)$ propagating with unit velocity along the positive x direction alone. It satisfies the wave equation

$$\phi_t + \phi_x = 0 \tag{58}$$

which can be checked by direct substitution. From here onwards, a subscript stands for the partial derivative with respect to it unless otherwise stated. Let us modify Eq.(58) by adding a term ϕ_{xxx}, i.e., consider

$$\phi_t + \phi_x + \phi_{xxx} = 0. \tag{59}$$

Such a term occurs in the Korteweg-de Vries equation which we shall examine in Lecture 4. If we take a plane wave of unit amplitude

$$\phi(x,t) = e^{-i(\omega t - kx)}. \tag{60}$$

and insert it into the above equation, we arrive at the following equation

$$\omega(k) = k - k^3 \tag{61}$$

which gives the frequency of the wave as a function of the wave number (or, equivalently, of the wavelength). Such a relation is called the dispersion relation. This allows us to obtain the following expressions for the phase velocity v_p and the group velocity v_g:

$$v_p = \frac{\omega}{k} = 1 - k^2, \quad v_g = \frac{\partial \omega}{\partial k} = 1 - 3k^2. \tag{62}$$

Equation (62) implies that waves with different wave numbers have different velocities. As a result a wave packet which, as we have seen in the last lecture, consists of a large number of monochromatic waves, will disperse as it propagates down the medium. Thus Eq.(59) represents the simplest form of dispersive wave equation.

If, instead, we add $-\phi_{xx}$ and consider

$$\phi_t + \phi_x - \phi_{xx} = 0 \tag{63}$$

and examine its plane wave solution, we arrive at the dispersion relation

$$\omega(k) = k - ik^2 \tag{64}$$

which, when inserted into the plane wave solution yields

$$\phi(x,t) = e^{-k^2t+ik(x-t)} = e^{-k^2t}\, e^{ik(x-t)}. \tag{65}$$

the full solution of the wave equation (63). It describes a wave that propagates with unit velocity for all k but this wave also decays exponentially for any real $k \neq 0$ as $t \to \infty$. This decay is referred to as dissipation. So the wave equation (63) represents a dissipative wave equation. Finally, we wish to examine the consequences of a nonlinear modification of the original wave equation. For this purpose, we consider the wave equation

$$\phi_t + (1+\phi)\phi_x = 0. \tag{66}$$

It is a nonlinear equation in the dependent variable ϕ. Linear wave equations are valid for relatively small amplitudes. In many real physical problems, physical characteristics of the medium, that determine the character of wave propagation in it, depend on the amplitude of the wave propagating through it. Under these circumstances, nonlinear wave equations, like the one written above, arise. The above form of nonlinearity has been considered keeping in view, once again, the Korteweg-de Vries equation. By the method of characteristics one obtains the general solution in the form

$$\phi(x,t) = f(x - (1+\phi)t) \tag{67}$$

where f is an arbitrary function of its argument. One can check, after a bit of algebra, that the function given by equation (67) does satisfy the wave equation (66). If we look at it carefully, we notice that given an initial profile $\phi(x,0) = f(x)$ the points that have higher amplitude will travel faster than those with smaller amplitude. As a result the wave front will steepen during propagation and ultimately the wave will break up. The spreading of a wave-hump at the sea-shore is an example of such a wave break-up.

It is clear that by making suitable assumptions in the underlying physical problem we might get an equation that is both nonlinear and dispersive or dissipative. The problem is then to solve this equation and physically interpret its solutions. Usually it is a tough task because general methods of solution of nonlinear partial or ordinary differential equations have not been as developed as they are for linear equations. In what follows we shall discuss some special nonlinear and dispersive equations that can be solved analytically exactly and which give rise to a very interesting class of solutions called solitons.

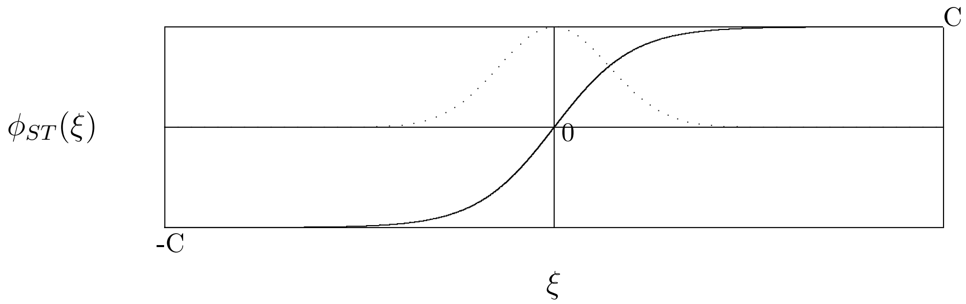

Fig.3: Two kinds of localization for solitary waves

Before we give a general definition of a soliton, we shall need the following concepts. As we have seen, given a wave equation, a travelling wave solution $\phi_T(x,t)$ is the one that depends on x and t only through $\xi = x - vt$, where v is a constnat.

Localized travelling wave solutions are called solitary waves and satisfy the following definition.

Solitary wave: A solitary wave $\phi_{ST}(\xi)$ is a travelling wave whose transition from one asymptotic state at $\xi \to -\infty$ to another at $\xi \to +\infty$ is essentially localized in ξ.

Two kinds of localization are possible as shown in Fig.3. In one case the function $\phi_{ST}(\xi)$ tends to zero as ξ tends to $\pm\infty$, while in the other case it tends to $\pm C$, respectively, where C is a constant. One might wonder, looking at the localization in the second case, whether such a solution is really localized. However, such localized solutions do occur in condensed matter physics and field theory. Since the derivative of this solution has the same localization as shown in the first case, the physical characteristics of the excitations represented by such solutions turn out to be finite and hence they are not distinguished from solutions of the first type.

Soliton: A soliton $\phi_s(x,t)$ is a solitary wave solution of a wave equation that asymptotically preserves its shape and velocity under collision with other solitary waves.

In other words, if the collision between solitary waves is elastic, the solitary waves are called solitons. It follows from this definition that a soliton is always a solitary wave, however, a solitary wave may not be a soliton. Hence, if we find a solitary wave solution of a nonlinear and dispersive wave equation, we still need to study its collision property to ascertain whether it is a soliton or not.

The simplest example of a soliton is a pulselike travelling wave solution of the linear dispersionless wave equation

$$\frac{\partial^2\phi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\phi(x,t)}{\partial t^2} = 0. \tag{68}$$

It sounds strange but it is true because there is nothing, no dispersion, no nonlinearity or dissipation to distort the pulse. It will travel in the medium without any change.

The above mentioned possibility of soliton formation is a highly ideal situation. No real medium satisfies that condition. Real media are dispersive, dissipative and nonlinear. Studies of pulse propagation in different media have established certain criteria, to be satisfied by the medium in which we study the phenomenon of wave propagation, for solitary wave type excitations to occur.

Linear dispersionless medium (solitary wave)	Linear medium with dispersion (No solitary wave)
Nonlinear dispersionless medium (No solitary wave)	Nonlinear medium with dispersion (solitary wave)

In a dispersive but linear medium solitary waves can not form because the pulse will broaden and disperse because of the reasons mentioned earlier. Similarly, a nonlinear medium without dispersion will lead to pulse break up as explained earlier. Only in a nonlinear and dispersive medium two seemingly opposite effects, dispersive broadening and nonlinear steepening, are balanced under appropriate conditions and solitary waves result.

In the next lecture, we shall consider certain nonlinear and dispersive wave equations that arise in different branches of physics and admit solitary wave solutions. Collision studies have shown that the majority of them happens to be solitons.

Lecture-4: Wave equations that admit solitary waves

The Korteweg-de Vries equation: The simplest nonlinear and dispersive wave equation that occurs, of course under certain assumptions, in various branches of physics is the so called Korteweg-de Vries equation

$$\phi_t + \alpha\, \phi\, \phi_x + \beta \phi_{xxx} = 0, \tag{69}$$

where the subscript x stands for the partial derivative of $\phi(x,t)$ with respect to x, α and β are constants, and ϕ stands for the deviation from the average value of a physical quantity which depends on the problem at hand. It is derived from the governing equations of irrotational two-dimensional motion of an incompressible inviscid fluid, bounded above by a free surface and below by a rigid horizontal plane, for small amplitude waves under specific boubdary conditions. The KdV equation is used to describe lossless propagation of shallow water waves, ion-acoustic waves and magnetohydrodynamic waves in plasmas, longitudinal waves in an elastic rod, pressure waves in a liquid-gas bubble mixture, internal gravity waves in a stratified fluid, waves in a rotating atmosphere (Rossby waves), anharmonic lattice, thermally excited phonon packets in low temperature nonlinear crystal etc.

We look for the solitary waves of the KdV equation in the form $\phi(x,t) = \phi(x - vt) \equiv \phi(\xi)$. Substitution into the equation (69) yields

$$(\alpha\phi - v)\phi_\xi + \beta\phi_{\xi\xi\xi} = \frac{\partial}{\partial\xi}\left(\frac{\alpha\phi^2}{2} - v\phi + \beta\phi_{\xi\xi}\right) = 0 \tag{70}$$

where the subscript ξ stands for the ordinary derivative of ϕ with respect to ξ. Integrating once we obtain

$$\frac{\alpha\phi^2}{2} - v\phi + \beta\phi_{\xi\xi} = C \tag{71}$$

where C is a constant. Since the solitary waves are localized

$$\lim_{|\xi|\to\infty} \phi = \lim_{|\xi|\to\infty} \phi_\xi = \lim_{|\xi|\to\infty} \phi_{\xi\xi} = 0, \tag{72}$$

the constant C is equal to zero. Multiplying Eq.(71) by ϕ_ξ (with C=0) and integrating once we obtain

$$\frac{d\phi}{d\xi} = \frac{\sqrt{\left(v - \frac{\alpha}{3}\phi\right)}}{\sqrt{\beta}}\ \phi, \tag{73}$$

where the constant of integration has again been put equal to zero in view of the conditions (72). Let us assume that the peak of the solitary wave is located at $\xi = 0$ and let its value be ϕ_0. Then

$$\frac{d\phi}{d\xi}|_{\xi=0} = 0. \tag{74}$$

Then Eq.(73) yields

$$v - \frac{\alpha}{3}\phi_0 = 0 \tag{75}$$

which leads to the following relationship between the peak amplitude and the velocity of the solitary wave

$$v = \frac{\alpha}{3}\phi_0. \tag{76}$$

Further, we get

$$\int_{\phi_0}^{\phi} \frac{dy}{\sqrt{\left(v - \frac{\alpha}{3}y\right)}\ y} = \frac{1}{\sqrt{\beta}}\int_0^{\xi} d\xi \tag{77}$$

Using the substitution

$$y = \frac{3v}{\alpha}\ sech^2 z, \tag{78}$$

and $sech^{-1}(1) = 0$ we obtain

$$-\frac{2}{\sqrt{v}}\int_0^{sech^{-1}(\sqrt{\alpha\phi/6v})} dz = \pm\frac{\xi}{\sqrt{\beta}}. \tag{79}$$

Taking into account that the solution is an even function, the choice of $\pm$ sign becomes redundant. As a result we arrive at the solution

$$\phi_{ST}(x,t) = \frac{3v}{\alpha}\ sech^2\left(\sqrt{\frac{v}{4\beta}}\ (x - vt)\right) = \phi_0\ sech^2\left(\sqrt{\frac{v}{4\beta}}\ (x - vt)\right). \tag{80}$$

The solution describes a hump travelling along the positive x direction. Collision studies show that the KdV solitary waves are actually solitons. Also, since the velocities of these solitons are proportional to their peak amplitudes, taller the soliton faster it moves. The width of the soliton is inversely proportional to the square root of its velocity. The sign of the soliton depends on the sign of the constant α. If the sign of α is negative, the solution describes a density depression travelling along the positive x axis.

The Nonlinear Schroedinger equation: The equation

$$i\phi_t + \phi_{xx} + \kappa|\phi|^2\phi = 0, \tag{81}$$

where κ is known as nonlinear Schroedinger equation. It is used to describe several phenomena in nonlinear optics, like, one dimensional self phase modulation of a monochromatic wave, stationary two dimensional self-focusing of a plane wave, self-trapping phenomena etc. It also describes the propagation of a heat pulse in a solid, Langmuir waves in plasmas and is closely related to the Ginsburg-Landau equation of superconductivity.

In nonlinear fiber optics it is used to study distortion free propagation of a laser pulse in an optical fiber. It is derived from Maxwell's equations under the so called slowly varying envelope approximation. For a laser pulse

$$E(z,t) = A(z,t)\, e^{i(kz-\omega t)} + c.c., \tag{82}$$

where $A(z,t)$ is the complex envelope amplitude of the pulse and c.c. stands for the complex conjugate, propagating along the z-axis, the nonlinear Schroedinger equation reads

$$iq_\xi + \frac{1}{2}q_{\tau\tau} + |q|^2 q = 0. \tag{83}$$

where q is the normalized complex amplitude of the electric field, ξ is the normalized distance of propagation along the fiber and τ is the normalized time measured in the frame moving with the group velocity of the pulse (see, for instance, references 3,6 and 7).

We look for the solitary wave solution in the following form,

$$q(\xi,\tau) = \sqrt{\psi(\xi,\tau)}e^{i\theta(\xi,\tau)}, \tag{84}$$

where the function ψ is assumed to be localized in τ and satisfies

$$\lim_{\tau\to\pm\infty} \psi(\xi,\tau) = \lim_{\tau\to\pm\infty} \frac{\partial\psi(\xi,\tau)}{\partial\tau} = 0. \tag{85}$$

Substituting for q from Eq.(84) into Eq.(83) and separating the real and imaginary parts we obtain

$$\frac{\partial \psi}{\partial \xi} + \frac{\partial}{\partial \tau}\left(\psi \frac{\partial \theta}{\partial \tau}\right) = 0 \tag{86}$$

and

$$-\theta_\xi + \frac{1}{4}\psi_{\tau\tau} - \frac{1}{8}\frac{\psi_\tau^2}{\psi^2} + \psi = 0. \tag{87}$$

Since Eq.(83) is for the frame moving with the group velocity of the pulse, the shape of the solitary wave be stationary in ξ. It means that $\partial\psi/\partial\xi = 0$, i.e., the function ψ does not depend ξ. If we take this into account we get from Eq.(86) that

$$\frac{\partial}{\partial \tau}\left(\psi \frac{\partial \theta}{\partial \tau}\right) = 0 \tag{88}$$

or

$$\psi \frac{\partial \theta}{\partial \tau} = c(\xi). \tag{89}$$

For a localized solution $c(\xi) = 0$ and we obtain

$$\theta = \theta(\xi) = \beta\xi + \theta_0, \tag{90}$$

where β is a constant and has the meaning of a nonlinear propagation constant shift and $\theta_0 = \theta(\xi = 0)$ stands for the initial value of the phase. If we set $\theta_0 = 0$ we obtain

$$\theta = \beta\xi, \quad \frac{\partial \theta}{\partial \xi} = \beta \quad \frac{\partial \theta}{\partial \tau} = 0. \tag{91}$$

Multiplying Eq.(87) by ψ_τ, using (91) and integrating once we obtain the following ordinary differential equation for ψ

$$\frac{d\psi}{d\tau} = \pm 2\psi\sqrt{2\beta - \psi}, \tag{92}$$

where the constant of integration has been put equal to zero in view of the localization condition (104). Let us assume that the solitary wave has its peak $q_0 = \sqrt{\psi_0}$ at $\tau = \tau_0$. Then we have $(d\psi_0/d\tau)|_{\tau_0} = 0$ and

$$\sqrt{\psi_0} = \sqrt{2\beta}. \tag{93}$$

This leads to

$$\int_{\psi_0}^{\psi} \frac{dy}{y\sqrt{2\beta - y}} = \pm\ 2(\tau - \tau_0), \tag{94}$$

The integral on the left-hand side, computed by the substitution $y = sech^2 z$, is a standard one and yields

$$\int_0^{sech^{-1}\sqrt{\frac{\psi}{2\beta}}} dz = (\tau - \tau_0). \tag{95}$$

Or,

$$\frac{1}{\sqrt{2\beta}}\ sech^{-1}\left(\frac{\sqrt{\psi}}{\sqrt{2\beta}}\right) = (\tau - \tau_0). \tag{96}$$

Thus the solitary wave solution of Eq.(83) is given by

$$q(\xi, \tau) = \sqrt{2\beta}\ sech\left(\sqrt{2\beta}\ (\tau - \tau_0)\right)\ e^{i\beta\xi} \tag{97}$$

or,

$$q(\xi, \tau) = q_0 sech\left(q_0(\tau - \tau_0)\right)\ e^{\frac{1}{2}\ iq_0^2\xi} \tag{98}$$

Note that the above solitary wave solutions are stationary bright solitary waves of Eq.(83). Since Eq.(83) is invariant under the Galilean transformation

$$q(\xi, \tau) \rightarrow e^{-i\frac{1}{2}\ v^2\xi + iv\tau}\ q(\xi, \tau - v\xi), \tag{99}$$

where v is the velocity of a Galilean frame, in a frame moving with velocity v with respect to the pulse, the solitary wave solutions of Eq.(83) are given by

$$q(\xi, \tau) = q_0 sech[q_0(\tau - v\xi)]\ e^{(-\frac{i}{2}v^2\xi + iv\tau + \frac{i}{2}q_0^2\xi)}. \tag{100}$$

Collision studies of these solitary waves show that they are solitons. Because the solitons of the nonlinear Schroedinger equation are robust under small perturbations, they are tipped to be used in long distance all-optical fiber-optic communication systems.

The sine-Gordon equation: The partial differential equation

$$\phi_{xx} - \phi_{tt} = sin\phi \tag{101}$$

is known as sine-Gordon equation. It arises in several branches in physics and has been used to describe the propagation of crystal dislocation, Bloch wall motion of magnetic crystals, propagation of magnetic flux on a Josephson line, unitary theory of elementary particles etc. Once again, note that the above equation has been written in the nondimensional form.

This equation has infinite number of trivial solutions, like $\phi_0 = 0, \pm 2\pi, \pm 4\pi, ...,$ which are called the vacuum solutions. Besides these, there are nontrivial solutions that interpolate between two consecutive vacuua. If, for instance, we take the vacua $\phi_0 = 0$ and $\phi_0 = 2\pi$, then the solution that satisfies

$$\lim_{x\to -\infty} \phi(x,t) = 0, \qquad \lim_{x\to +\infty} \phi(x,t) = 2\pi \tag{102}$$

is called the kink solution, while the solution that satisfies

$$\lim_{x\to +\infty} \phi(x,t) = 0, \qquad \lim_{x\to -\infty} \phi(x,t) = 2\pi \tag{103}$$

is called the antikink solution. Let us determine this solution. As earlier we look for the solitary wave solution of the sine-Gordon equation in the form $\phi(x,t) = f(x-vt)$. Substitution into the differential equation yields

$$f'' = \gamma^2 \, sinf, \quad \gamma^2 = \frac{1}{(1-v^2)}. \tag{104}$$

The first integral of this equation is

$$\frac{f'^2}{2} = A - \gamma^2 \, cosf \tag{105}$$

where A is the constant of integration and is determined from the condition that $\lim_{|x|\to\infty} f(x,t) = \phi_0$. It turns out to be γ^2. For $v^2 < 1$ the solution is obtained from

$$\int_{f_0}^{f} \frac{df}{\sqrt{1-cosf}} = \int_{f_0}^{f} \frac{df}{\sqrt{2}sin(f/2)} = \pm\sqrt{2}(\xi - \xi_0) = \pm\sqrt{2}(x - x_0 - vt), \tag{106}$$

where x_0 is the constant of integration. Or,

$$ln\left(tan\frac{f}{4}\right) = \pm 2\gamma\,(x - x_0 - vt). \tag{107}$$

The soliton solution is thus given by

$$\phi(x,t) = 4tan^{-1}\left(e^{\pm 2\gamma\,(x - x_0 - vt)}\right) \tag{108}$$

These solutions correspond to a rotation in ϕ by 2π as x goes from $-\infty$ to $+\infty$. The plus sign corresponds to the positive sense of rotation, whereas the minus sign to the negative sense of rotation. The former is called a kink and the latter an antikink.

Concluding, we would like to note that soliton solutions are generic, and, although real systems often contain mechanisms (dissipative forces and other additional perturbative effects depending on the problem at hand) that may destroy the exact soliton behaviour, solitons are very useful for a starting point for analysis. In a way, they play the same role in nonlinear physics as the one played by the simple harmonic oscillators in linear physics. More over, perturbative methods, which consider perturbations around the soliton solution, have also been worked out to calculate the response of the soliton to dissipative forces and other external perturbations. One may say that a very useful branch of physics - Soliton Physics - has firmly established itself.

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