NIUS Lecture Notes

On

Basic Quantum Mechanics

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1 Introduction

The history of the modern, programmable computer starts with the mechanical difference engine of Charles Babbage. Since then, computers have undergone many changes, from analog to digital, from mechanical systems to vacuum tubes and eventually transistors, integrated circuits. The drive has always been to perform computational tasks quickly, and more economically by making computers smaller and more energy-efficient. The paradigm in computer technology for the last few decades has been the integrated circuit - a large number of transistors packed onto a small silicon chip. Breakthroughs in microprocessor design and semiconductor technology have allowed us to pack in more and more transistors on a chip of a given size. As a result, computing power has doubled nearly every decade since the 1950's, as observed and predicted by Gordon Moore in 1965.

However, this trend cannot continue indefinitely, simply because of the limitations of physical laws. Our transistors certainly cannot be smaller than atoms. Thus, there will be an upper limit to the number of transistors per area we can pack into a silicon wafer. In fact, classical computation would probably become impossible much before our devices hit the limit of one atom. And quantum mechanics is fundamentally different from the laws of classical physics that govern the operation of present-day computers.
As we know, the elementary unit of classical computation and information theory is the bit, a quantity that can take one of two values – 0 and 1. The quantum mechanical analogue of such an object is, however, not restricted only to these two possibilities; in fact, it can exist in infinite possible states that are, for lack of better word, 'in-between' 0 and 1. It is only after we measure the system that we can say definitively that it is in 0 or in 1. Also, we cannot say beforehand whether we will get 0 or 1 when we measure such an 'in-between' state.

Thus classical computational methods, algorithms etc. are likely to fail in the quantum domain. We need new and innovative methods to utilize quantum physics for the purpose of computation and information processing. Remarkably, it has been shown that algorithms designed using quantum mechanical rules can in fact achieve greater efficiency and speed over their classical counterparts. Problems whose solutions required an exponential number of steps using classical algorithms can now, at least theoretically, be solved in a few steps by quantum algorithms. Likewise, quantum mechanics is being

exploited in related areas of information processing, such as for secure communication, cryptography and even teleportation. With its promise of revolutionary changes in our technology, quantum information and quantum computation is now one of the most active areas of scientific research.

In this chapter, we shall familiarize ourselves with basic quantum mechanics, which is a prerequisite for studying quantum computation. We shall state and discuss the postulates of quantum mechanics, highlighting their difference from classical physics. The necessary mathematical formalism and notation is introduced alongside. The mathematical language of quantum mechanics is linear algebra. Thus, a familiarity with matrix operations proves sufficient for most quantum mechanical calculations.

2 Postulates of quantum mechanics

Let us first state without preamble the postulates of quantum mechanics. We review them in detail in the subsequent sections.

1. The states of physical systems are represented by vectors in Hilbert space.
2. Measurable or observable quantities are represented by Hermitian operators.
3. The evolution of a closed quantum system occurs through the action of unitary operators.
4. When a measurement is performed, the state is projected onto one of the eigenstates of the corresponding Hermitian operator, with a probability proportional to the inner product between the state vector before the measurement and the eigenstate it is projected onto, yielding the eigenvalue as the measurement outcome.

3 State vectors and Hilbert space

3.1 Dimension and basis of a vector space

The first postulate states that an n-dimensional quantum state can be represented by a state vector or a state function, which is a point in an n-dimensional linear vector space, defined over the complex field $\mathbb{C}$.
A set of n vectors ϕ_k is said to be linearly independent if and only if the equation

$$\sum_{k=1}^{n} x_k \phi_k = 0 \tag{1}$$

has the unique solution

$$x_1 = x_2 = ... = x_n = 0, \tag{2}$$

where x_k's are scalar coefficients. The vector space being defined over $\mathbb{C}$ means that x_k's are, in general, complex numbers. If there exists a set of scalars, not all of which are zero,

such that one of the vectors in this vector space can be expressed as the linear combination of others,

$$\phi = \sum_{k=1}^{n} x_k \phi_k, \tag{3}$$

then the set ϕ_i is said to be linearly dependant. The basis of the vector space consists of a set of the maximum possible numbers of linearly independent vectors belonging to that space. These vectors $\phi_1, \phi_2 ... \phi_n$ to be denoted by ϕ_i are called basis vectors. Generally for the sake of convenience, in quantum mechanics, these vectors are chosen to be orthonormal to each other, i.e., their scalar products satisfy the relation,

$$(\phi_i, \phi_j) = \delta_{ij} \tag{4}$$

A basis is said to be orthonormal if the basis vectors are orthonormal to one another. It is complete if the basis vectors span the entire space. An example of linear vector space is the Euclidian three dimensional space. Here the basis vectors are $\hat{i}$, $\hat{j}$ and $\hat{k}$ and can be written in matrix form as

$$\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5}$$

Any arbitrary vector in this space can be expressed as $\overrightarrow{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, which in this column matrix notation will be given as,

$$\overrightarrow{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{6}$$

where, the coefficients, A_x, A_y and A_z are real numbers. However, for a quantum state in general they will be complex numbers. Such a vector space defined over $\mathbb{C}$ is called a Hilbert space. We will encounter applications of the properties of this space in the next chapter, when we deal with qubits. In Dirac's bra-ket notation, the wavefunction ψ and is represented as $|\psi\rangle$(ket) and its complex conjugate ψ^* is represented as $\langle\psi|$ (bra). Kets are elements of the Hilbert space for every ket there is a unique bra and vice versa. The bra element belong to $\mathbb{H}^*$ which is the dual space of the Hilbert space $\mathbb{H}$ of the ket vectors. The orthonormality condition in bra-ket notation is denoted by, $\langle\Psi|\Phi\rangle$=0 (orthogonality), $\langle\Psi|\Psi\rangle$=1 and $\langle\Phi|\Phi\rangle = 1$ (normalization).

3.2 Properties of Hilbert Space

Let us summarize again the properties of Hilbert space:

1. The states (or vectors) are commutative under addition,
2. Their addition is also associative,
3. The inner product of two states exists and the norm of a vector is positive definite,
4. The vector space is a complete.

4 Observables and measurement

4.1 Hermitian Operators

In quantum mechanics, observables are postulated to be Hermitian operators mapping a Hilbert space $\mathcal{H}$ onto itself. Mathematically, a matrix H is Hermitian if $H^\dagger = H$ where $H^\dagger$ denotes the transpose of the complex conjugate of H, i.e., a Hermitian matrix is self-adjoint. The other important properties of Hermitian operators are:

1. The eigenvalues of a Hermitian operator are real.

2. The eigenvectors or eigenstates of a Hermitian operator span the Hilbert space, which means that each observable generates a complete orthonormal basis.

4.2 Measurement

The measurement of an observable in quantum mechanics will give an eigenvalue of the corresponding Hermitian operator as the measurement outcome. This works out to be quite simple when the state vector being measured is in an eigenstate of the measurement operator. Then, the outcome is merely *that* particular eigenvalue. That is, if we measure an observable, given by the Hermitian $\hat{A}$, on a state $|\psi_a\rangle$ where $\hat{A}|\psi_a\rangle = \lambda_a|\psi_a\rangle$, the measurement outcome is λ_a.
A general state vector will not be in a particular eigenstate of the operator we are measuring; it will be in some superposition of the possible eigenstates, e.g., $|\psi\rangle = \sum_{i=1}^{n} c_i|\phi_i\rangle$, where the eigenvalues of the $|\phi_i$ are given as λ_i, $\hat{A}|\phi_i\rangle = \lambda_i|\phi_i\rangle$.
A measurement of $\hat{A}$ on this state will probabilistically give as outcome one of the eigenvalues. The probability of obtaining an outcome λ_j is given by $P(\lambda_j) \propto |\langle\phi_j|\psi\rangle|^2 = |c_j|^2$.
The constant of proportionality can be determined by demanding that the net probability be 1. $\sum_i P(\lambda_i) = 1$ gives,

$$P(\lambda_j) = \frac{|\langle\phi_j|\psi\rangle|^2}{\sum_i |\langle\phi_i|\psi\rangle|^2} = \frac{|\langle\phi_j|\psi\rangle|^2}{\langle\psi|\psi\rangle} \tag{7}$$

Equivalently, we can ensure that the norm of the state vector ($|\psi|$) is 1, by redefining the state vector as

$$|\psi\rangle \to \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}. \tag{8}$$

Formally, this is know as *normalizing* the wavefunction.
The probabilistic nature of the outcome of a quantum mechanical observation is at odds with the deterministic view of nature posited by classical mechanics. In classical physics, if we know the dynamics of a system and are given sufficient initial conditions, we can uniquely specify the properties of the system at any future instant of time. Here, we see that specifying the initial quantum state of a system generally proves insufficient for predicting the outcome of a measurement.

As a result, in quantum mechanics we usually talk about the *expectation value* of an observable, i.e., its average over large number of observations. This is given by

$$\langle \hat{A} \rangle = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle} \tag{9}$$

Now, suppose we have performed a measurement of $\hat{A}$ on a quantum system in the state $|\psi\rangle$ and obtained some λ_j as outcome. It is found that subsequent observation of $\hat{A}$ on this system (provided it does not interact with any other system) keeps yielding λ_i as the outcome. This follows naturally if we initially had $|\psi\rangle$ to begin with, but the observation holds true for any general quantum state. The implication is that the act of measuring $\hat{A}$ has changed the system from $|\psi\rangle$ to some $|\phi_j\rangle$, which is the eigenvector corresponding to the measurement outcome λ_j. In other words, the measurement has caused a *collapse of the state vector*, projecting it onto one of the eigenvectors of the observable.

In this context, we define the *projection operator*,

$$\hat{P}_i = |\chi_i\rangle\langle\chi_i| \tag{10}$$

where $\{|\chi_i\rangle\}$ is an orthonormal basis. Any observable can be expressed in terms of projection operators as

$$\hat{A} = \sum_i \lambda_i \hat{P}_i \tag{11}$$

where $\hat{P}_i$ projects to the eigenstate $|\phi_i\rangle$. This is known as *spectral decomposition*.
The measurement we have mentioned can be written as the appropriate projector $\hat{P}_j = |\phi_j\rangle\langle\phi_j|$ acting on $|\psi\rangle$.
Also, the identity matrix can be written as,

$$\hat{I} = \sum_i \hat{P}_i = \sum_i |\phi_i\rangle\langle\phi_i| \tag{12}$$

This relation is known as the *resolution of identity* and proves very useful in different calculations. For example, we can write the expectation value of observable $\hat{A}$ as (up to a normalization factor)

$$\langle \hat{A} \rangle \propto \langle \psi | \hat{A}\hat{I} | \psi \rangle = \sum_i \langle \psi | \hat{A} | \phi_i \rangle \langle \phi_i | \psi \rangle = \sum_i \lambda_i |\langle \phi_i | \psi \rangle|^2 \tag{13}$$

which is simply the average of eigenvalues weighted with the corresponding probabilities. This particular form was found because we resolved the identity matrix in the eigenbasis of $\hat{A}$. In any other basis the expression will be

$$\langle \hat{A} \rangle \propto \sum_i |\langle \chi_i | \psi \rangle|^2 \langle \chi_i | \hat{A} | \chi_i \rangle \tag{14}$$

In the preceding discussion, we have confined ourselves to operators with *discrete spectrum*, i.e., operators whose eigenvalues are finitely spaced (as indicated by the use of

summation). Examples of such operators are the three Pauli matrices, which correspond to spin along three different directions,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

All three matrices have two distinct eigenvalues, ± 1 . For σ_z, the eigenvectors are,

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{16}$$

The spectral decomposition for σ_z is therefore

$$\sigma_z = 1|\uparrow\rangle\langle\uparrow| - 1|\downarrow\rangle\langle\downarrow| \tag{17}$$

This is an example of a spectrum which is discrete and finite. One can also think of operators which have discrete spectrum but infinite spectrum, e.g., the energy levels of a simple harmonic oscillator, which are eigenstates of the corresponding Hamiltonian.
There also exist operators whose eigenvalues are infinitesimally spaced, and which therefore generate uncountably infinite eigenbasis, i.e., we cannot label their eigenvectors as $|\phi_1\rangle, |\phi_2\rangle, ..., |\phi_n\rangle ...$ etc. Position and momentum operators fall into this class. Such operators are said to have a *continuous spectrum*. The Hilbert space spanned by their eigenbasis is infinite-dimensional.
For these operators, the summations we have used previously must be replaced with integrals.
For instance, in the one-dimensional position basis $\{|x\rangle\}$ (defined by $\hat{X}|x\rangle = x|x\rangle$), the resolution of identity becomes

$$I = \int dx |x\rangle\langle x| \tag{18}$$

The orthonormality condition is defined through the Dirac delta function in the case of continuous systems,

$$\langle x|y\rangle = \delta(x - y) \tag{19}$$

The expectation value of an observable $\hat{A}$ takes the following form in the position basis $\{|x\rangle\}$,

$$\langle\hat{A}\rangle = \frac{1}{\langle\psi|\psi\rangle}\int dx \langle\psi|\hat{A}|\psi\rangle = \frac{1}{\mathcal{N}}\int dx \langle\psi|x\rangle\langle x|\hat{A}|x\rangle\langle x|\psi\rangle = \frac{1}{\mathcal{N}}\int \psi^*(x)\hat{A}(x)\psi(x)dx \tag{20}$$

Here $\psi(x)$ denotes the position-space representation of the state vector $|\psi\rangle$, and its complex conjugate is denoted by $\psi^*(x)$. The normalization $\mathcal{N}$ can be calculated as

$$\mathcal{N} = \int \psi^*(x)\psi(x)dx \tag{21}$$

and $\hat{A}(x)$ is the position-space representation of the observable $\hat{A}$.
Analogously, we can calculate the above quantities in the same manner using the momentum basis $\{|p\rangle\}$, which is formed by the state vectors that obey

$$\hat{P}|p\rangle = p|p\rangle \tag{22}$$

The passage from position to momentum basis (and vice versa) is through the Fourier transform,

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int e^{-ipx} \psi(x) dx, \;\; \psi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \phi(p) dp \tag{23}$$

The position operator ($\hat{X}$) in the position basis is simply the coordinate x, where whereas the momentum operator is given by $\hat{P} = -i\hbar \frac{\partial}{\partial x}$.
These representations provide a *correspondence* between the macroscopic classical picture and the microscopic quantum description, the variables x and p being replaced by the corresponding operators. Here, we note that a variable $f \rightarrow f(x,p)$, which is a function of both x and p, will be replaced in the quantum theory by the corresponding function of operators, $\hat{F} = f(\hat{X}, \hat{P})$.
For example, the angular momentum operator, $\vec{l} = \vec{r} \times \vec{p}$ is represented by the quantum observable,

$$\hat{L} = \hat{\vec{R}} \times \hat{\vec{P}} \tag{24}$$

where $\hat{\vec{R}} = \hat{X}\hat{i} + \hat{Y}\hat{j} + \hat{Z}\hat{k}$ and $\hat{\vec{P}} = \hat{P}_x\hat{i} + \hat{P}_y\hat{j} + \hat{P}_z\hat{k}$.
In the position basis, this can be written component-wise

$$\hat{L}_i(x_i) = \sum_{j,k=1}^{3} \epsilon_{ijk} x_j \frac{\partial}{\partial x_k} \tag{25}$$

where ϵ_{ijk} is the full antisymmetric tensor.

4.3 The Hamiltonian

A canonical example of an observable used extensively in quantum mechanics is the total energy operator or the Hamiltonian, which is the sum of kinetic energy (K) and the potential energy (V). Thus,

$$\hat{H} = \hat{T} + \hat{V} \tag{26}$$

The eigenvalue equation for this operator is known as the time independent Schrödinger equation and is given by,

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{27}$$

where, "E" is the energy eigenvalue. Upon solving this equation, one obtains a set of eigenvalues and corresponding eigenvectors $\{E_n, |\Psi_n\rangle\}$, such that the $\{|\Psi_n\rangle\}$'s form an orthonormal set.

4.4 Non-commutativity

The commutator of two matrices is defined as,

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{28}$$

Previously, we have seen that observables in quantum mechanics are represented by Hermitian matrices, and measurement probabilistically projects a quantum state to one of the

eigenstates of that operator. Now, in general, two observables $\hat{A}$ and $\hat{B}$ need not commute, i.e.,

$$[\hat{A}, \hat{B}] \neq 0. \tag{29}$$

This, is a problem from the point of view of classical physics. Two linear operators can have a common set of eigenvectors only if they commute with each other - this is a well-known result from linear algebra. Non-existence of a common eigenbasis for two Hermitian operators implies that the corresponding observables cannot be simultaneously measured.
This is quite easy to see. Suppose $\hat{A}$ and $\hat{B}$ are two observables which do not commute. A measurement of $\hat{A}$ on a system will project it onto an eigenstate of $\hat{A}$. Now this eigenstate is some superposition of the eigenstates of $\hat{B}$. Thus, performing a measurement of $\hat{B}$ now will cause the state to change, collapsing it to an eigenvector of $\hat{B}$.
Momentum and position are prime examples of two such non-commuting observables. From the definitions of these operators, it is easy to verify that,

$$[\hat{X}, \hat{P}] = i\hbar \tag{30}$$

Therefore, the position and momentum observables cannot have a common set of eigenfunctions.
The eigenstates of position in position space are the Dirac delta functions, $\delta(x)$. Taking Fourier transform, we see that its momentum space representation is $\phi(p) = e^{-ipx}$.
$\phi(p)$ is a plane wave - essentially a superposition of all possible values of momentum. Similarly, a wavefunction localized to a point in momentum space will be a plane wave in position space. This leads us to conclude that it is impossible for a particle to have well-defined values of position and momentum simultaneously.

5 Time evolution

5.1 The Schrödinger equation

Quantum mechanics enables us to calculate the time evolution of a dynamical system provided initial state of the system is known and the Hamiltonian for the system is specified. The Hamiltonian must include all potentials and interactions that the system is subject to. In the *Schrödinger picture* of quantum mechanics, the dynamics is governed by the time evolution of the state, which results from the solution of the time-dependent Schrödinger equation,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \tag{31}$$

where $|\psi(t)\rangle$ denotes the state of the system at any one time t, $\hat{H}$ is the Hamiltonian and $\hbar$ is the reduced Planck constant. The Hamiltonian is time-dependent in general.
The time-evolved state can thus be written as

$$|\psi(t)\rangle = e^{-i\int \hat{H}(t)t/\hbar} |\psi(0)\rangle \tag{32}$$

The simplest example of time-evolution is that of the eigenstates of the Hamiltonian, i.e., the energy eigenstates. Substituting $|\psi(0)\rangle = |E\rangle$ in the (32) we get,

$$|\psi(t)\rangle = e^{-i\hat{H}(t)t/\hbar}|E\rangle = e^{-iEt/\hbar}|E\rangle$$

So, an energy eigenstate changes only by an overall phase-factor through time evolution. In particular, as an overall phase-factor cannot affect measurement outcomes in quantum mechanics, its energy remains unchanged.

5.2 Heisenberg Picture

In the Schrödinger picture, the state vectors evolve in time. The Heisenberg picture presents an alternate way of looking at time-evolution - the state vectors remain fixed while the observables, i.e, the operators become time-dependent. The evolution of the operators is governed by the Heisenberg equation,

$$\frac{dU(t)}{dt} = -\frac{i}{\hbar}[U(t), H]$$

Note that the operator $U(t) = e^{-iHt/\hbar}$ is unitary, i.e, $U^\dagger = U^{-1}$, Any evolution of a quantum system can be expressed as a unitary operation, unless a measurement takes place somewhere. Unitary evolution is completely deterministic once the initial conditions and the system Hamiltonian are specified. It is measurements that break this determinism through probabilistic collapse of the wavefunction.

Summary

In this chapter we have studied the basic mathematical formulation of quantum mechanics. As we have seen, it is a linear theory and has a probabilistic character. The wavefunction plays a central role in quantum mechanics, containing all information about the particle(s) it describes. Quantum mechanical entities have both wave and particle nature at the same time. The properties of a quantum mechanical system are obtained by applying suitable operators on these wavefunctions. Mathematically, these operations are similar to matrix algebra. As a consequence, two operations need not commute, thereby making non-commutativity a key feature of quantum mechanics.
An observation, in which the information from the state can be obtained by a detector, plays an important role in quantum mechanics. A closed quantum system undergoes unitary evolution but observation leads to a "collapse of wavefunction" to "eigenstates" of the operator. This intrusive nature of measurement is a very curious feature of quantum mechanics.

6 Problems

1. Calculate the eigenvalues and eigenvectors of pauli matrices
   $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. Suppose a quantum system is in the state, $|\psi\rangle = \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix} + \frac{2}{5}\begin{pmatrix}0\\1\end{pmatrix}$
and we perform a measurement of the observable $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.
What are the possible measurement outcomes and what are the respective probabilities? Calculate the expectation value of S_z.

3. Suppose that in the previous problem, we obtained the outcome $s_z = \frac{\hbar}{2}$. What will be the measurement outcome of $S_x = \frac{\hbar}{2}\sigma_x$ on the resultant state?

4. Find the unitary matrices that diagonalize the Pauli matrices.

5. Prove that for an arbitrary square matrix A, $detA = e^{trace(lnA)}$.

6. Show the eigenvalues of a hermitian matrix are always real and eigenvectors corresponding to different eigenvalues are orthogonal.

7. Show that the modulus of eigenvalues of a unitary matrix is always 1. (This ensures that the norm of a state vector is preserved under unitary evolution.)

8. Show that $[X, P] = i\hbar$

9. Verify the following commutator identities:
   (a) $[A, B + C] = [A, B] + A, C$
   (b) $[A, BC] = B[A, C] + [A, B]C$
   (c) $[AB, C] = A[B, C] + [A, C]B$
   (d) The Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$

10. Suppose that you are given an initial state $|\psi(0)$ such that $H|\psi(0)\rangle = E|\psi(0)\rangle$, i.e., $|\psi(0)\rangle$ is an eigenstate of the Hamiltonian. Fine out the time-evolved state $|\psi(t)$ using the Schrödinger equation.

11. Consider a quantum state of an electron consisting of a magnetic moment, $\overrightarrow{\mu} = \gamma\,\hbar\,\overrightarrow{\sigma}$ evolving under a magnetic field. The Hamiltonian, H under a static magnetic field B applied along the x-axis is given by,

$$H = -\frac{1}{2}\hbar\,\omega\,\sigma_x$$

where, $\omega = \gamma\,B$ and γ is the gyromagnetic ratio.

   (a) If initially the electron is in the state $|0\rangle = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, find out the minimum time necessary to flip the spin from $|\uparrow\rangle$ to $|\downarrow\rangle$. Find also the time necessary to take the initial state, $|\uparrow\rangle$ to an arbitrary state, $\phi = \frac{|\uparrow\rangle + i|\downarrow\rangle}{\sqrt{2}}$. Why this state cannot be distinguished from the state $|\uparrow\rangle$.
   (b) Calculate the expectation value of $\overrightarrow{\mu}$ in an arbitrary state after a time t since the evolution has started. Also give a physical interpretation of the result.

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